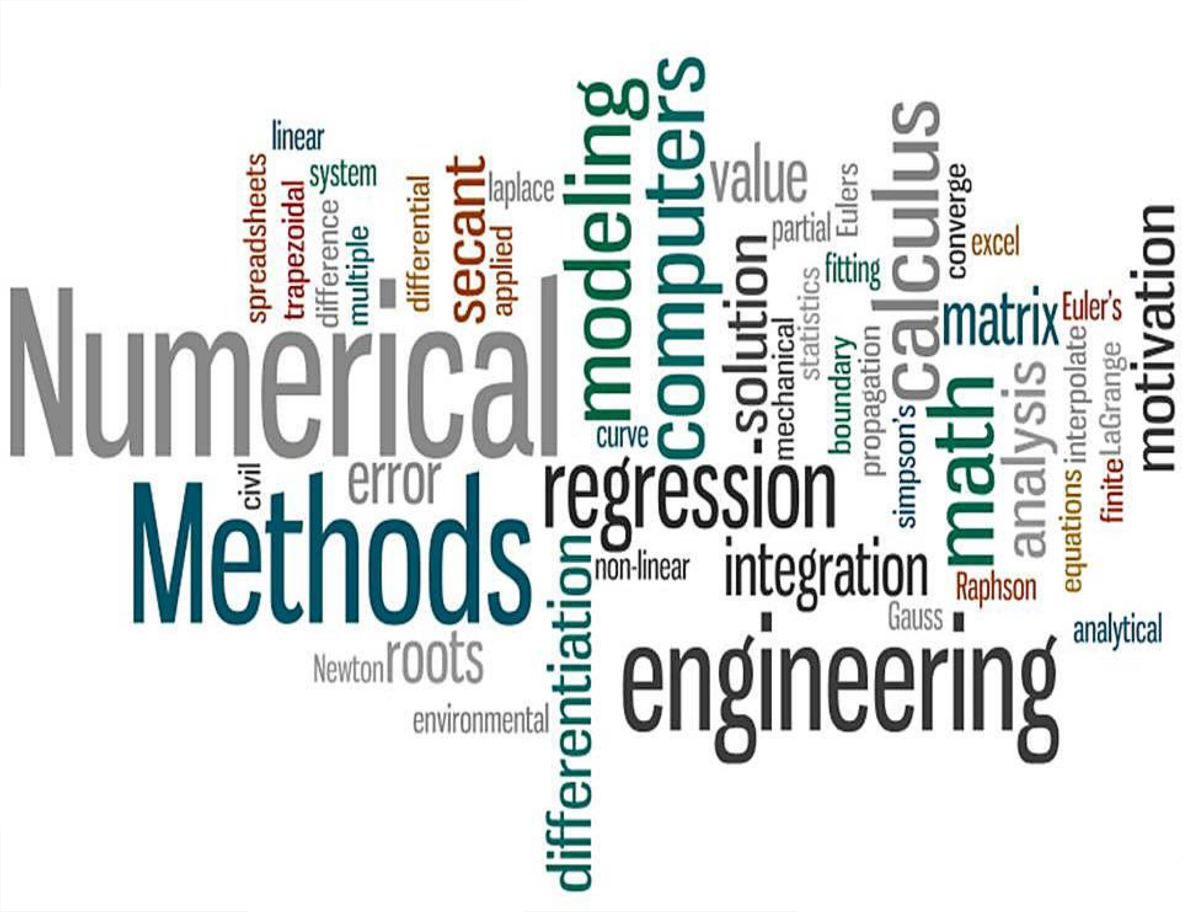
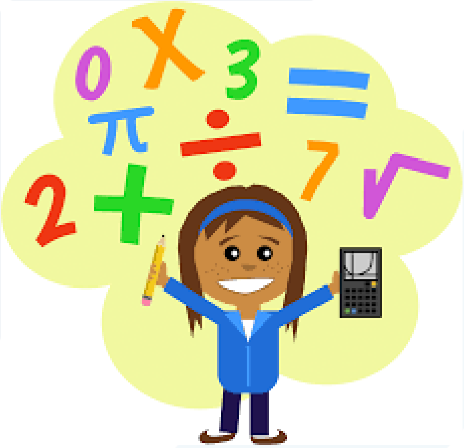
Programming and Numerical Analysis



What Are Numerical Methods

- Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations {+,-,*,/} that can then be performed by a computer.

الطرائق العددية هي تقنيات يتم من خلالها صياغة المسائل الرياضية بحيث يمكن حلها بالعمليات الحسابية وبالتالي يمكن حلها باستخدام الكمبيوتر.



Motivation

- **Although there are many kinds of numerical methods, they have one common characteristic: they invariably involve large numbers of tedious arithmetic calculations.**

- **على الرغم من وجود العديد من أنواع الطرائق العددية ، إلا أن لها خاصية مشتركة واحدة: فهي تنطوي دائماً على أعداد كبيرة من العمليات الحسابية المملة.**

- **It is little wonder that with the development of fast, efficient digital computers, the role of numerical methods in engineering problem solving has increased dramatically in recent years.**

- **لا عجب أنه ومع تطور أجهزة الكمبيوتر الرقمية السريعة والفعالة، زاد دور الطرائق العددية في حل المشكلات الهندسية بشكل كبير في السنوات الأخيرة.**

Non-Computer Methods

- **Solutions were derived for some problems using analytical, or exact, methods.**

تم اشتقاق حلول بعض المسائل باستخدام طرائق تحليلية أو دقيقة. في كثير من الأحيان، كانت هذه الحلول مفيدة وقدمت فهماً أعمق لسلوك بعض الأنظمة. ومع ذلك، فإن اشتقاق الحلول التحليلية متاح لفئة محدودة فقط من المسائل. تشمل تلك التي يمكن تقريبها باستخدام النماذج الخطية وتلك التي لديها شكل جيومتري بسيط وعدد مجاهيل منخفض. وبالتالي، فإن الحلول التحليلية غير عملية لأن معظم المسائل الحقيقية غير خطية وتنطوي على أشكال وعمليات معقدة.

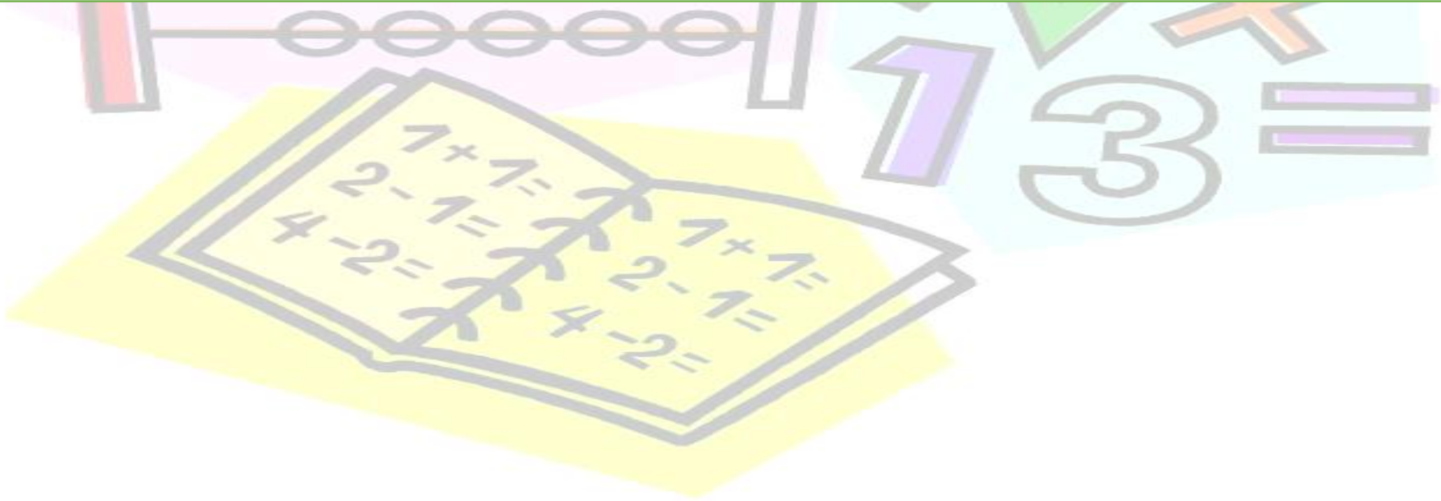
- **Graphical solutions were used to characterize the behavior of systems.**

تم استخدام الحلول الرسومية لتوصيف سلوك الأنظمة. عادة ما تأخذ هذه الحلول الرسومية شكل مخططات بيانية أو nomographs. على الرغم من أنه يمكن استخدام التقنيات الرسومية في كثير من الأحيان لحل المشكلات المعقدة، إلا أن النتائج ليست دقيقة بشكل كاف. علاوة على ذلك، فإن الحلول الرسومية (بدون مساعدة أجهزة الكمبيوتر) ستكون مملة ومتعبة للغاية وصعبة التنفيذ. أخيراً، غالباً ما تقتصر التقنيات الرسومية على المشكلات التي يمكن تمثيلها باستخدام ثلاثة أبعاد أو أقل.

Non-Computer Methods

- **Calculators and slide rules were used to implement numerical methods manually.**

تم استخدام الآلة الحاسبة والمسطرة الحاسبة، لتنفيذ الطرائق العددية يدوياً. على الرغم من أنه من الناحية النظرية، يجب أن تكون هذه الأساليب مناسبة تماماً لحل المشكلات المعقدة إلا أنها في الو اقع تواجه صعوبات متعددة. فالحسابات اليدوية بطيئة ومملة. علاوة على ذلك، فإن النتائج المنسجمة تكون مخادعة (بعيدة المنال) بسبب الأخطاء البسيطة التي تظهر عند تنفيذ العديد من المهام اليدوية.



Pre-Computer Era

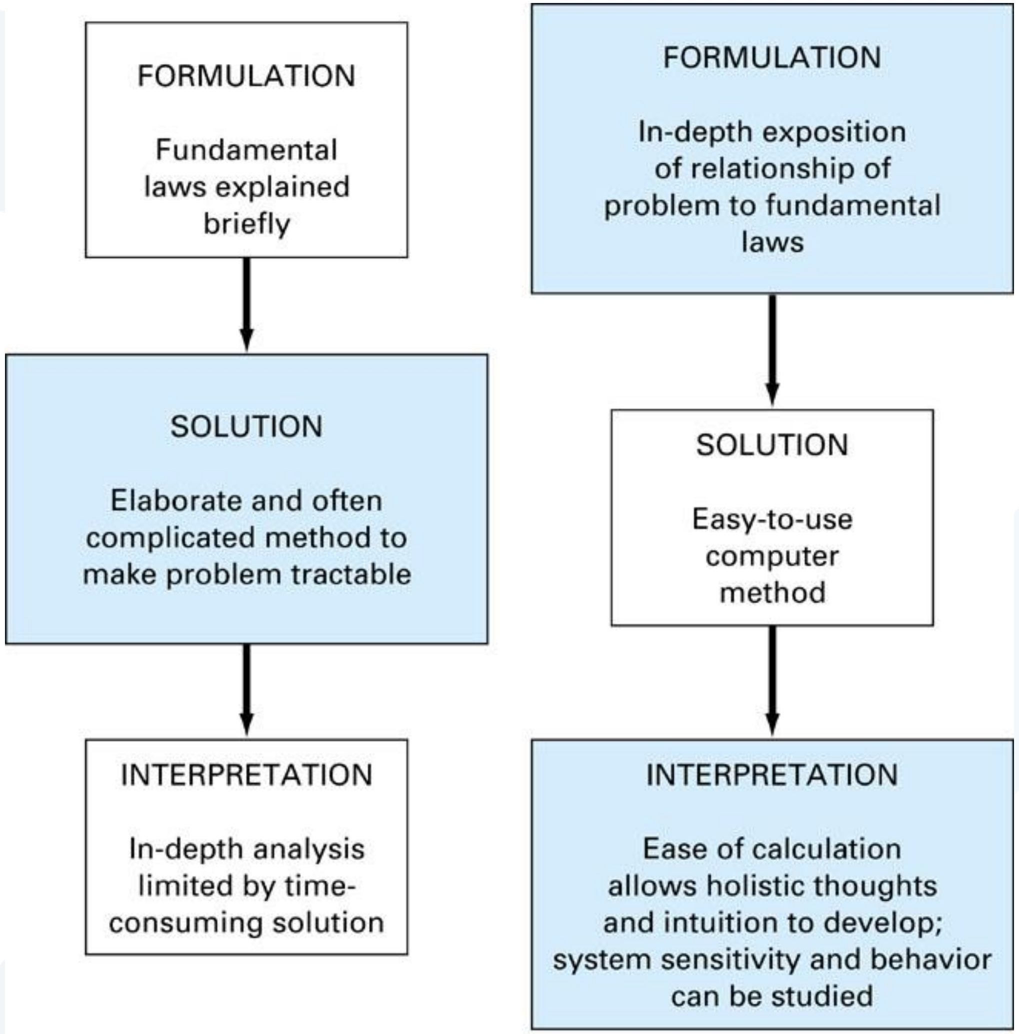
During the pre-computer era, significant amounts of energy were expended on the solution technique itself, rather than on problem definition and interpretation

خلال عصر ما قبل الكمبيوتر، تم بذل الكثير من الجهد على تقنية الحل نفسها، بدلاً من تعريف المشكلة وتفسيرها.

Computer Era

- Today, computers and numerical methods provide an alternative for such complicated calculations.
- Computer can be used to obtain solutions directly.
- Numerical methods represent alternatives that greatly enlarge your capabilities to confront and solve problems.
- As a result, more time is available for the use of your creative skills.
- More emphasis on problem formulation and solution interpretation and the incorporation of total system, or “holistic,” awareness.

Pre-Computer vs. Computer Era



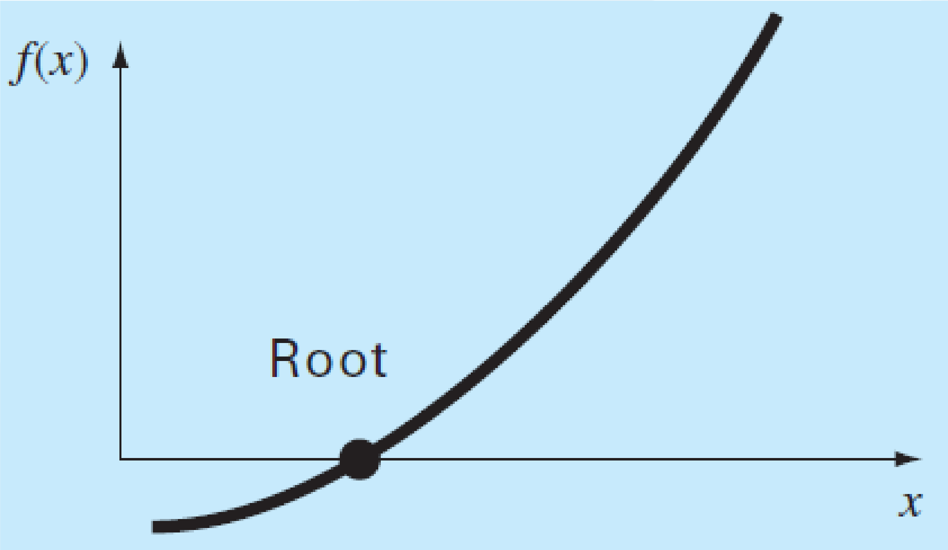
The Most Mathematical Areas in Numerical Method

1. Roots of Equations
2. Systems of Linear Algebraic Equations
3. Optimization
4. Curve Fitting
5. Integration
6. Ordinary Differential Equations
7. Partial Differential Equations

The Most Mathematical Areas in Numerical Method

- **Roots of equations:** concerns with finding the value of a variable that satisfies a single **nonlinear** equation – especial valuable in engineering design where it is often impossible to explicitly solve design equations of parameters.

Solve $f(x) = 0$ for x.



The Most Mathematical Areas in Numerical Method

- **Roots of equations:** concerns with finding the value of a variable that satisfies a single **nonlinear** equation – especial valuable in engineering design where it is often impossible to explicitly solve design equations of parameters.

Some simple equations can be solved analytically:

$$x^2 + 4x + 3 = 0$$

$$\text{Analytic solution } roots = \frac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$$

$$x = -1 \textbf{ and } x = -3$$

Many other equations have no analytical solution:

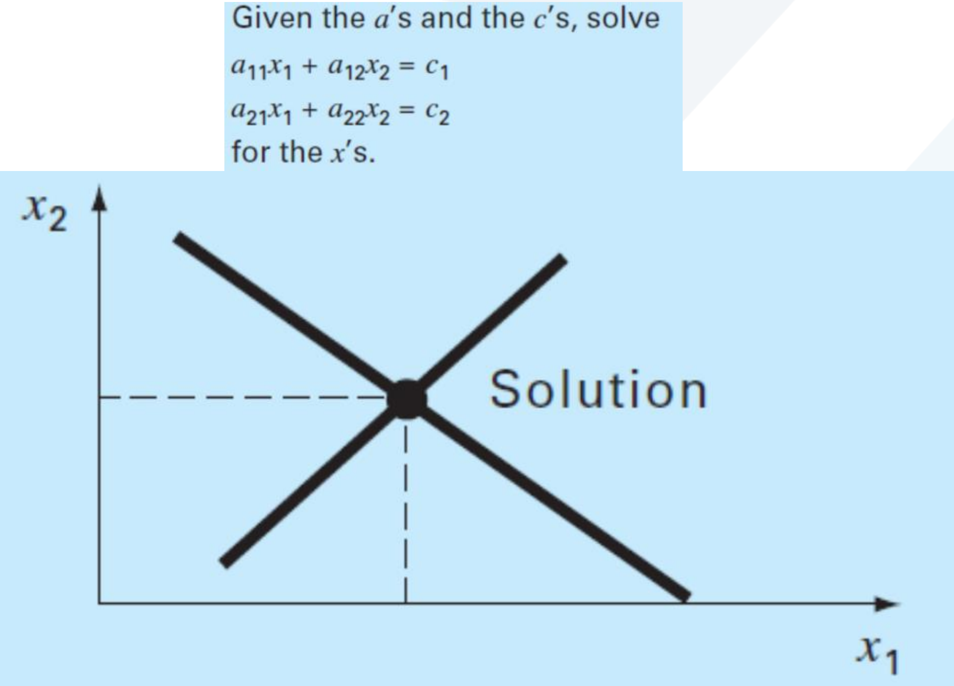
$$\left.\begin{aligned} x^9 - 2x^2 + 5 = 0 \\ x = e^{-x} \end{aligned}\right\} \text{No analytic solution}$$

The Most Mathematical Areas in Numerical Method

Methods for Solving Nonlinear Equations			
Graphical Method	Bisection Method	Fixed Point Iteration Method	Newton Raphson Method

The Most Mathematical Areas in Numerical Method

- **Systems of linear equations:** a set of values is sought that simultaneously satisfies a set of linear algebraic equations. They arise in all disciplines of engineering, e.g., structure, electric circuits, fluid networks; also in curve fitting and differential equations.



The Most Mathematical Areas in Numerical Method

- **Systems of linear equations:** a set of values is sought that simultaneously satisfies a set of linear algebraic equations. They arise in all disciplines of engineering, e.g., structure, electric circuits, fluid networks; also in curve fitting and differential equations.

$x_1 + x_2 = 3$
$x_1 + 2x_2 = 5$

We can solve it as:

$x_1 = 3 - x_2, \qquad 3 - x_2 + 2x_2 = 5$
$\Rightarrow x_2 = 2, \qquad x_1 = 3 - 2 = 1$

What to do if we have 1000 equations in 1000 unknowns.

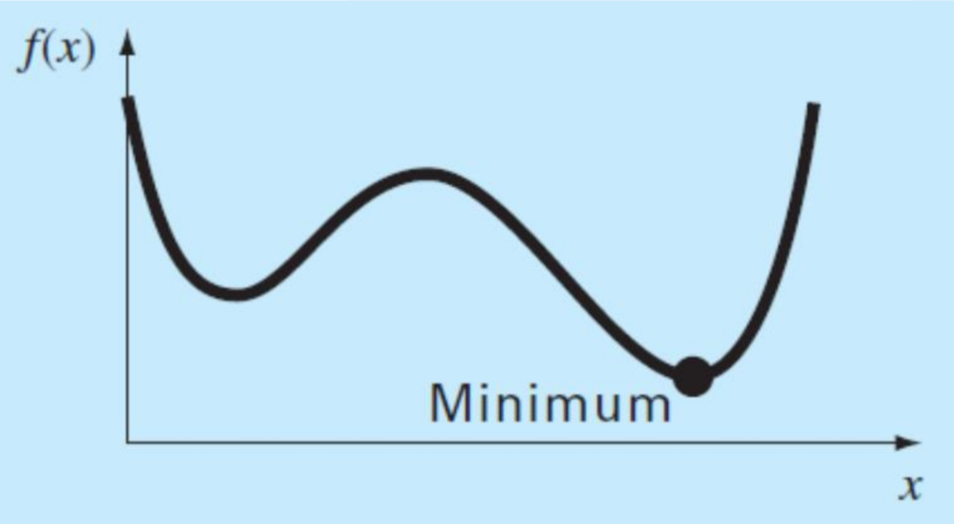
The Most Mathematical Areas in Numerical Method

Methods for Solving Systems of Linear Equations		
Graphical Solution	NAIVE GAUSS ELIMINATION	Gaussian Elimination with Scaled Partial Pivoting

The Most Mathematical Areas in Numerical Method

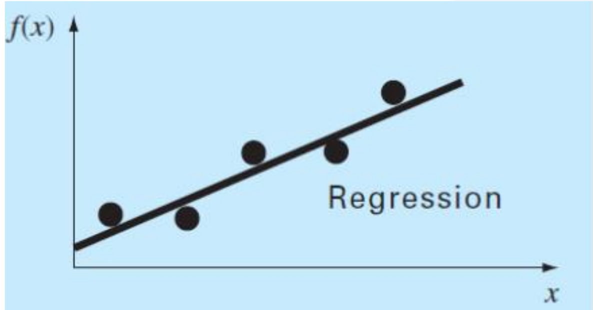
- **Optimization:** determine a value or values of an independent variable that correspond to a "best" or optimal value of a function. It occurs routinely in engineering contexts.

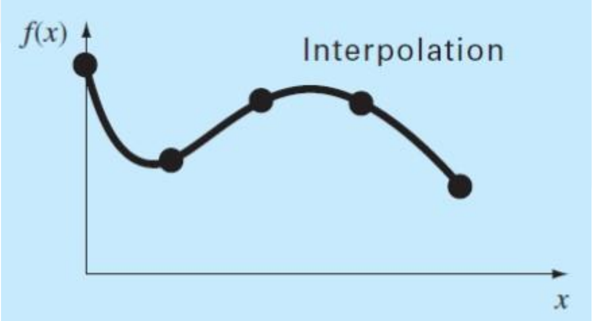
Determine x that gives optimum $f(x)$.



The Most Mathematical Areas in Numerical Method

- **Curve Fitting:** to fit curves to data points. Two types: regression and interpolation.



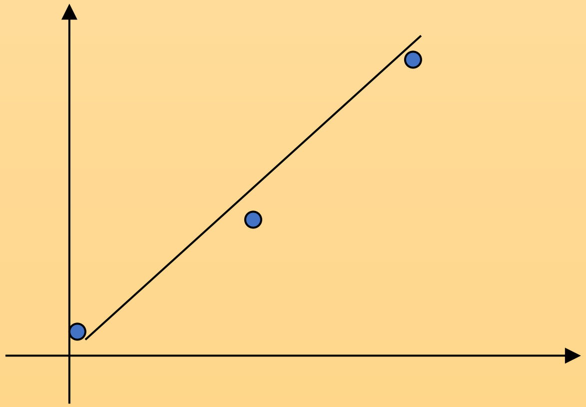


The Most Mathematical Areas in Numerical Method

- **Curve Fitting:** to fit curves to data points. Two types: **regression** and interpolation. Experimental results are often of the first type.

- Given a set of data:

x	0	1	2
y	0.5	10.3	21.3



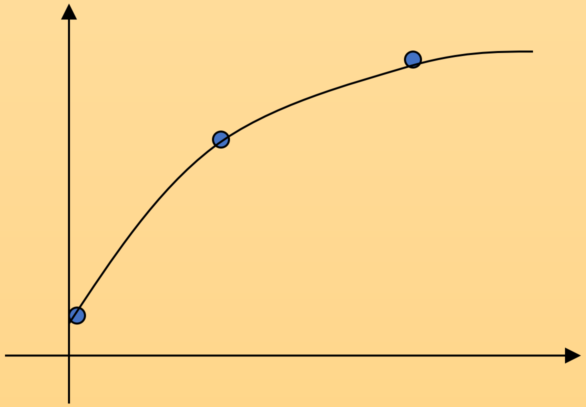
- Select a curve that best fits the data. One choice is to find the curve so that the sum of the square of the error is minimized.

The Most Mathematical Areas in Numerical Method

- **Curve Fitting:** to fit curves to data points. Two types: regression and interpolation. Experimental results are often of the first type.

- **Given a set of data:**

x_i	0	1	2
y_i	0.5	10.3	15.3



- **Find a polynomial *P(x)* whose graph passes through all tabulated points.**

$$y_i = P(x_i) \ \ if \ \ x_i \ \text{is in the table}$$

The Most Mathematical Areas in Numerical Method

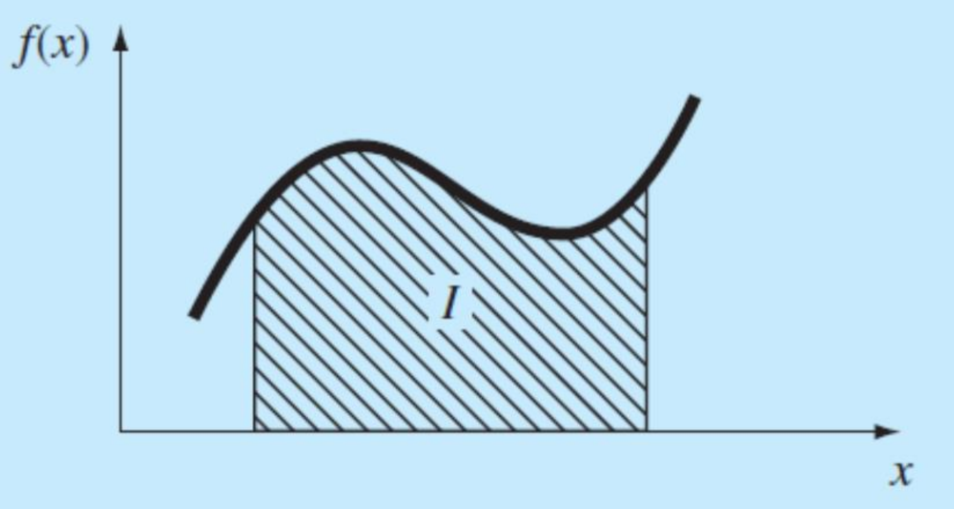
Methods for Curve Fitting	
Least Squares Regression	**Interpolation** • Newton's Polynomials Interpolation • Lagrange Interpolation

The Most Mathematical Areas in Numerical Method

- **Integration:** determination of the area or volume under a curve or a surface. It has many applications in engineering practice, such as.

$I = \int_a^b f(x)\, dx$

Find the area under the curve.



The Most Mathematical Areas in Numerical Method

- **Integration:** determination of the area or volume under a curve or a surface. It has many applications in engineering practice, such as.

Some functions can be integrated analytically:

$$\int_1^3 x dx = \frac{1}{2}x^2\Big|_1^3 = \frac{9}{2} - \frac{1}{2} = 4$$

But many functions have no analytical solutions:

$$\int_0^a e^{-x^2}\, dx = ?$$

The Most Mathematical Areas in Numerical Method

Methods for Numerical Integration

Trapezoidal Method	Simpson's Method

The Most Mathematical Areas in Numerical Method

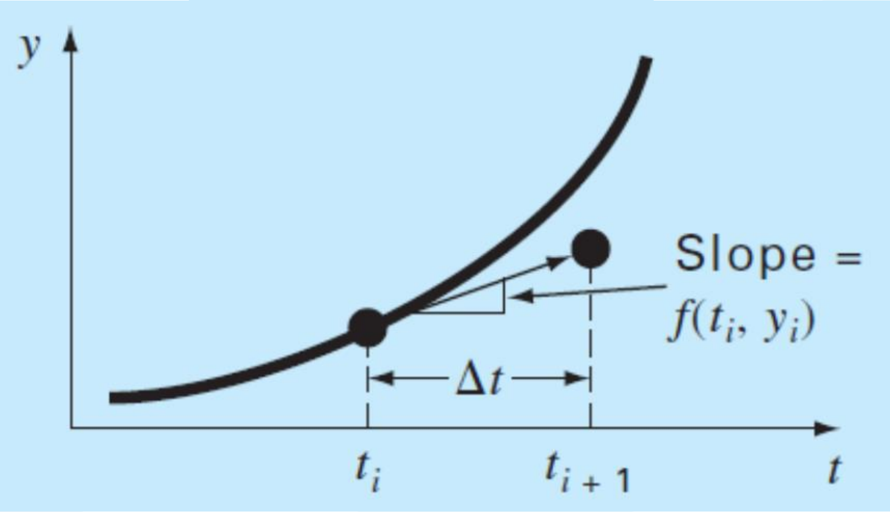
- **Ordinary Differential Equations:** very important in engineering practice, because many physical laws are couched in terms of the rate of change of a quantity rather than the magnitude of the quantity itself, such as ....

Given

$$\frac{dy}{dt} \simeq \frac{\Delta y}{\Delta t} = f(t, y)$$

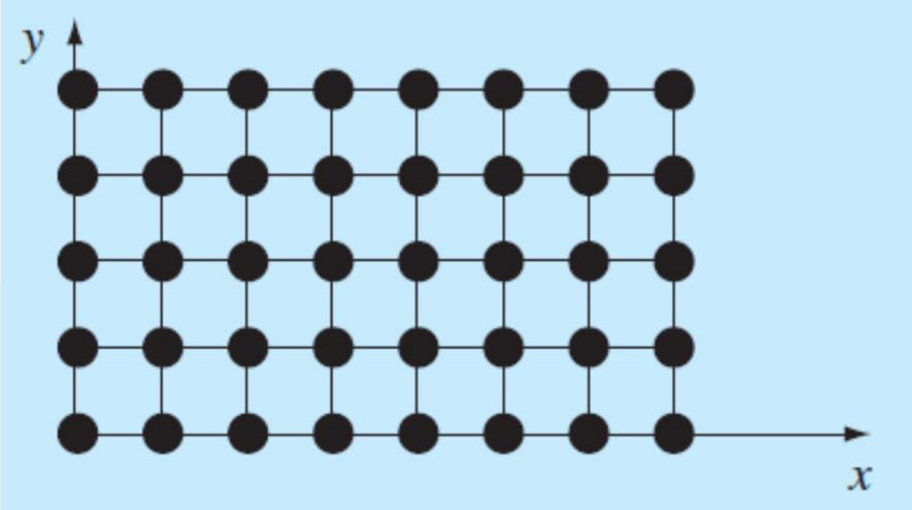
solve for y as a function of t.

$$y_{i+1} = y_i + f(t_i, y_i)\,\Delta t$$



The Most Mathematical Areas in Numerical Method

- **Partial Differential Equations:** used to characterize engineering systems where the behavior of a physical quantity is couched in terms of the rate of change with respect to two or more independent variables. Examples: steady-state distribution of temperature of a heated plate (two spatial dimensions) or the time-variable temperature of a heated rod (time and one spatial dimension).



Given

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

solve for u as a function of x and y

Numerical Methods and Engineering Practice

Since the late 1940s the widespread availability of digital computers has led to a veritable explosion in the use and development of numerical methods.

At first, this growth was somewhat limited by the cost of access to large mainframe computers.

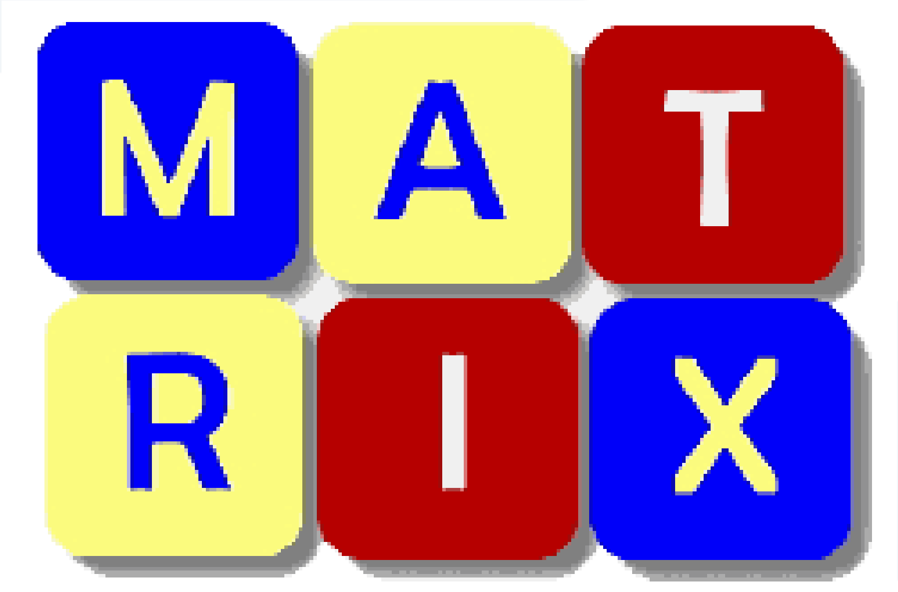
The recent evolution of inexpensive personal computers has given us ready access to powerful computational capabilities.

Why You Need to Learn Numerical Methods?

- Numerical methods are extremely powerful problem-solving tools.
- During your career, you may often need to use commercial computer programs (canned programs) that involve numerical methods. You need to know the basic theory of numerical methods in order to be a better user.
- You will often encounter problems that cannot be solved by existing canned programs; you must write your own program of numerical methods.
- Numerical methods are an efficient vehicle for learning to use computers.
- Numerical methods provide a good opportunity for you to reinforce your understanding of mathematics.

You need that in your life as an engineer or a scientist

Mathematical Background



VECTORS

Vector : a one dimensional array of numbers

Examples :

row vector $\begin{bmatrix} 1 & 4 & 2 \end{bmatrix}$ column vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$Identity\ \ vectors\ \ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

MATRECIS

Matrix : a two dimensional array of numbers

Examples :

zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

diagonal $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$, Tridiagonal $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

MATRECIS

Examples :

$$\text{symmetric}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix}, \quad \text{upper triangular}\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Adding and Multiplying Matrices

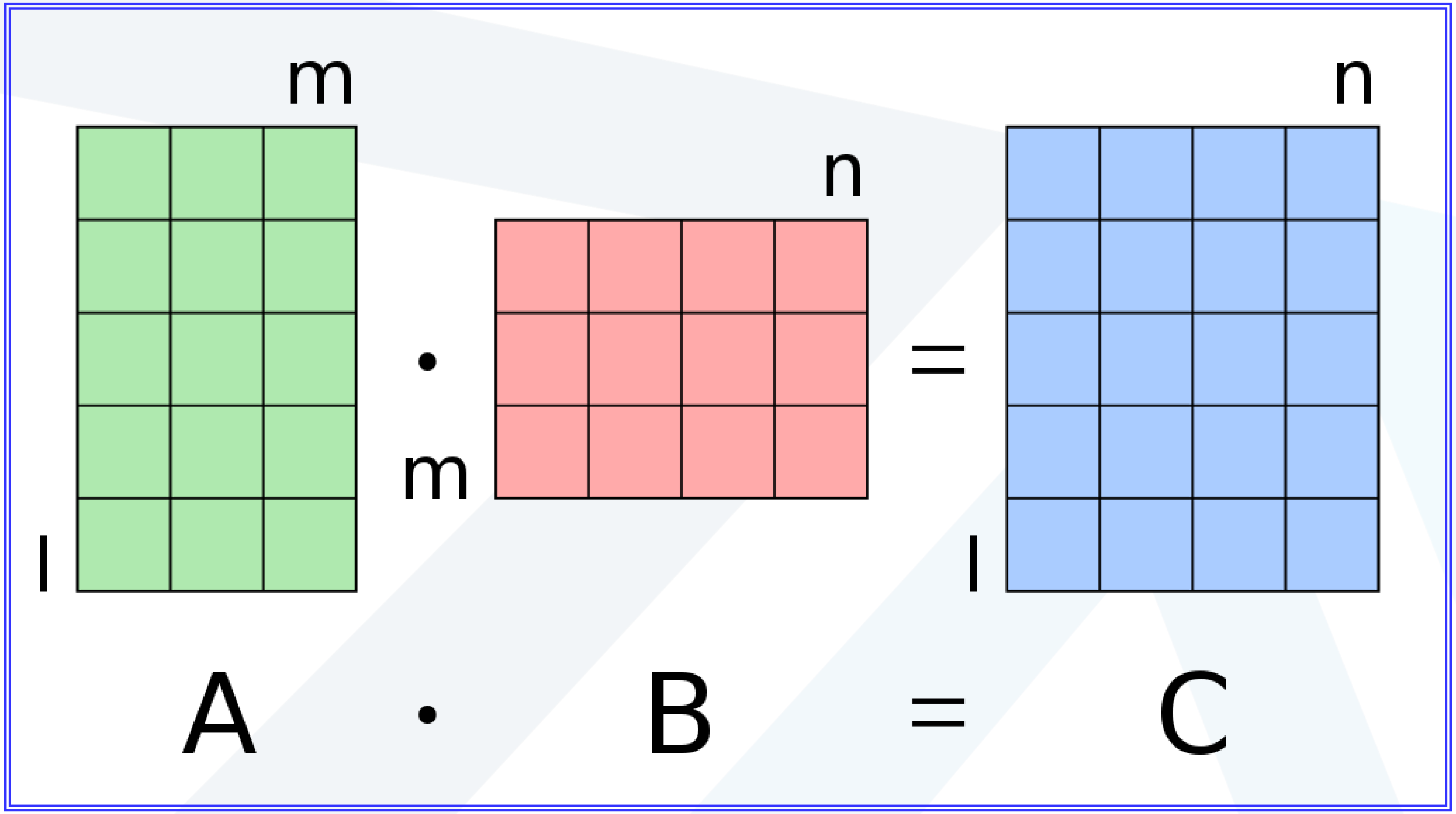
The addition of two matrices A and B

* Defined only if they have the same size
* $C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$

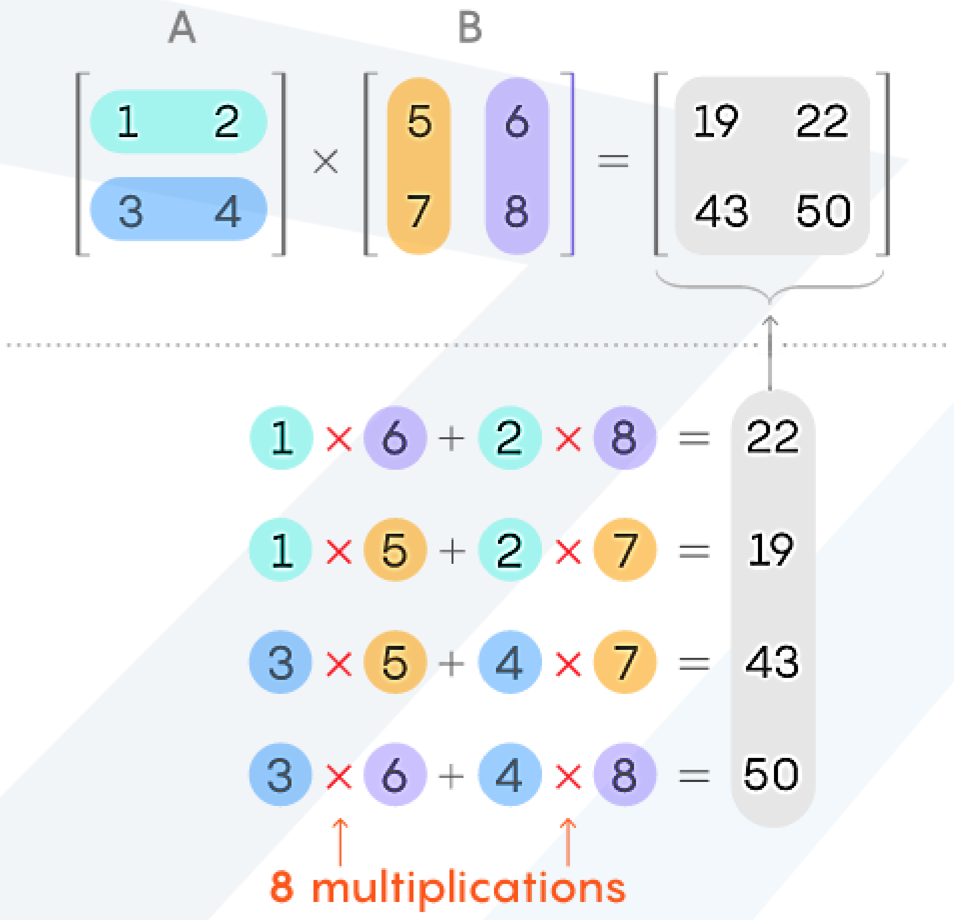
Multiplication of two matrices $A(n \times m)$ and $B(p \times q)$

* The product $C = AB$ is defined only if $m = p$
* $C = A\,B \Leftrightarrow c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \quad \forall i, j$

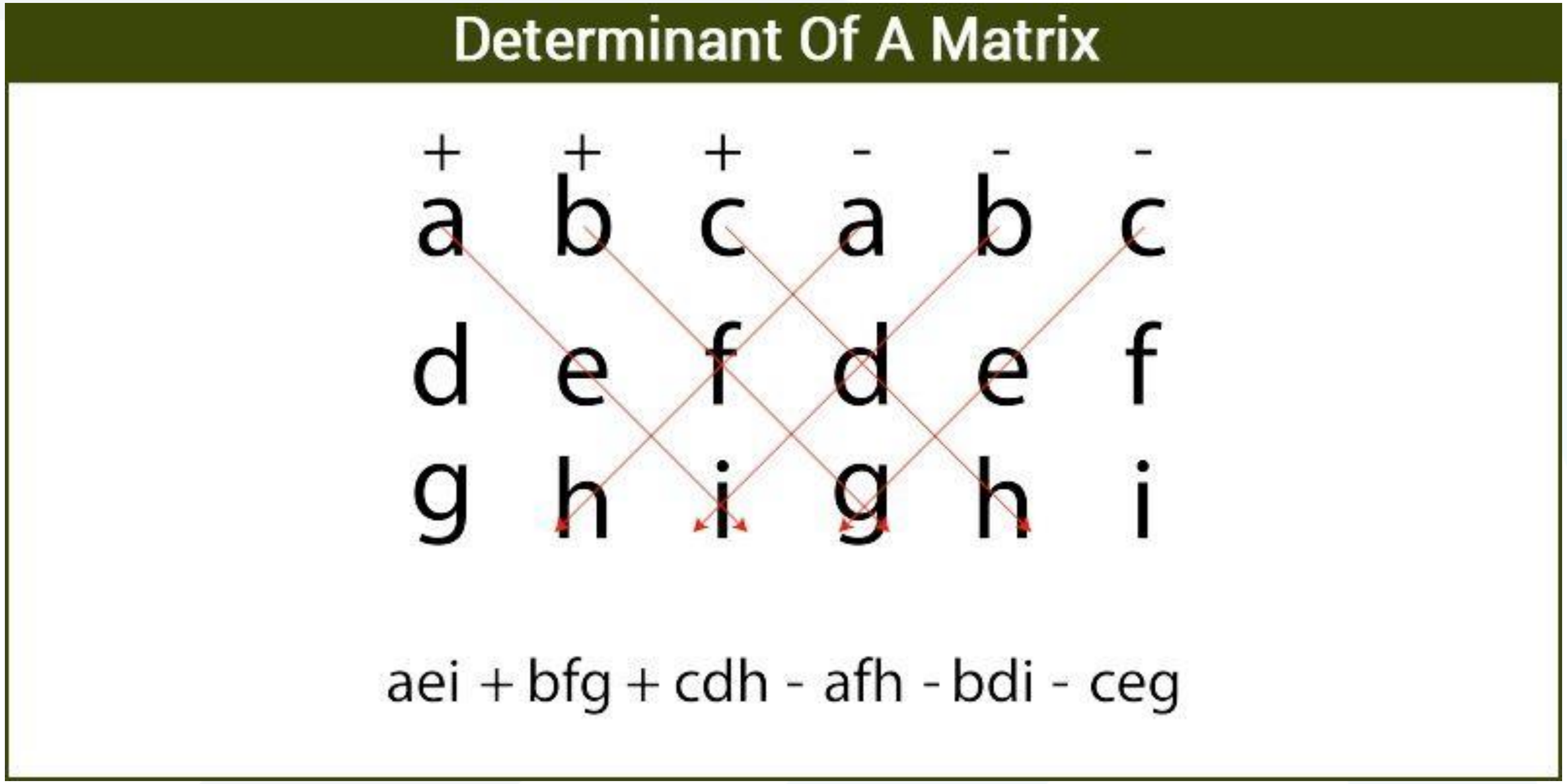
Multiplying Matrices



Multiplying Matrices



DETERMINANT of MATRECIS



DETERMINANT of MATRECIS

Defined for square matrices only

Examples :

$$\det\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix} = 2\begin{vmatrix} 0 & 5 \\ 5 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 5 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix}$$

$$= 2(-25) - 1(12+5) - 1(15-0) = -82$$

Inverse of Matrices

Inverse of 2x2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- A^{-1}: Inverse of A
- $ad - bc$: Determinant of A
- $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$: Adjoint of A

Note: A^{-1} exists only when $ad - bc \neq 0$